Numerical Simulations in Cosmology I

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Abstract

The purpose of these lectures is to give a short introduction into a very vast field of numerical simulations for cosmological applications. I focus on major features of the simulations: the equations, main numerical techniques, effects of resolution, and methods of halo identification.

1. Introduction

Numerical simulations play a very significant role in cosmology. It all started in 70s with simple N-body problems solved using primitive N-body codes with few hundred particles. Later the code which we now call Particle-Particle code (direct summation of all two-body forces) was polished and brought to the state-of-art [1]. Already those early efforts brought some very valuable fruits. In 1970 Peebles [2] studied collapse of a cloud of particles as a model of cluster formation. The model had 300 points initially distributed within a sphere with no initial velocities. After the collapse and virialization the system looked like a cluster of galaxies. Those early simulations of cluster formation, though producing cluster-like objects, signaled the first problem – simple model of initially isolated cloud (top-hat model) results in the density profile of the cluster which is way too steep (slope -3 -4) as compared with real clusters. The problem was addressed by Gunn & Gott [3], who introduced a notion of secondary infall in an effort to solve the problem. Another keystone work of those times is the paper by White [4], who studied collapse of 700 particles with different masses. It was shown that if one distributes the mass of a cluster to individual galaxies (=points), two-body relaxation will result in mass segregation which is not compatible with observed cluster. This

was another manifestation of dark matter in clusters. This time it was shown that the dark matter of a cluster is not in galaxies. Survival of substructures in galaxy clusters was another problem addressed in the paper. It is quite remarkable that 20 years later we are still facing the same two problems. For numerical models (at least some of them) it is the excessive mass segregation, and, thus, a wrong structure of objects due to insufficient number of particles. For cluster dynamics it is the amount of substructure (buzzwords: "merging", "hydrostatic equilibrium").

First cosmological simulations (growth and collapse of initially small fluctuations in expanding Universe) were done at about the same time (mid 70s). An example: Aarseth, Gott & Turner [5] studied the evolution of fluctuations using 5000 particles moving in expanding sphere. The particles were initially placed at random within the sphere (thus, the flat $n = 0$ spectrum of initial fluctuations). The sphere was needed to handle boundary conditions: if a particle happens to hit the sphere it is reflected back like a ball in a billiard. At that time the correlation function completely dominated cosmological landscape. Simple hierarchical clustering arguments [6] indicated that in order to reproduce the observed slope of the correlation function of galaxies $\gamma = -1.77$, the spectrum of initial fluctuations should be flat. Results of N-body simulations seems to confirm the conclusion. Another result, which followed from the same arguments was used to defeat a cosmological scenario proposed by Zeldovich [7], which was called at that time "adiabatical fluctuations" and later resurfaced under the name of the Hot Dark Matter. Because the Zeldovich model suggested very distinct scale in the initial spectrum of fluctuations (almost no fluctuations on small scales), the argument was that it should not produce a scale-free power-law for the correlation function. As we know now, both arguments were wrong. And it was found thanks to numerical simulations. Efstathiou & Eastwood [8] using much better code (one of the first results with Particle-Particle-Particle-Mesh (P^3M) code) with 20,000 particles showed that the model with the flat spectrum (initial random distribution) fails to produce power-law correlation function. Klypin & Shandarin [9] showed that the Hot-Dark-Matter (HDM) model predicts the power-law for the correlation

function. The HDM model is dead anyway (it does not have enough high-z objects), but the history of the model is really amazing: it was killed for sure so many times ... because of wrong reasons.

Generation of initial condition with given amplitude and spectrum of fluctuations was a problem for some time. The only correctly simulated spectrum was the flat spectrum which was generated by randomly distributing particles. In order to generate fluctuations with power spectrum, say $P(k) \propto k^{-1}$, Aarseth, Gott & Turner [5] placed particles along rods. Formally, it generates the spectrum, but the distribution has nothing to do with cosmological fluctuations. As far as I know, the papers of Doroshkevich *et al.* [10] for two dimensions and Klypin & Shandarin [9] in three dimensions were the first where initial conditions were generated using the Zeldovich [7] approximation – the method which since then was used to generate initial conditions.

Starting mid 80s the field of numerical simulations is blooming: new numerical techniques were invented, old ones were perfected, many publications (and, occasionally, results) are based on numerical modeling. To large extend, this have changed our way of doing cosmology. Instead of questionable assumptions and waving-hands arguments, we have tools of testing our hypothesis and models. As an example, I mention two analytical approximations which were validated by numerical simulations. The importance of both approximations is difficult to overestimate. The first is the Zeldovich approximation, which paved the way of understanding the large-scale structure of the galaxy distribution. The second is the Press-Schechter [11] approximation, which gives the number of objects formed at different scales at different epochs. Both approximations cannot be formally proved. The Zeldovich approximation formally is not applicable for hierarchical clustering. It must start with smooth perturbations (truncated spectrum). Nevertheless, numerical simulations have shown that even for the hierarchical clustering the approximation can be used with appropriate filtering of initial spectrum (e.g. [12,13,14]). The Press-Schechter approximation is also difficult to justify without numerical simulations. It operates with initial spectrum and linear theory,

but then (very long jump in assumptions) it predicts the number of objects at very nonlinear stage. Because it is not based on any realistic theory of nonlinear evolution (we just do not have one), it was an ingenious, but a wild guess. If anything, the approximation is based on a simple spherical top-hat model. But simulations show that objects do not form in this way – they are formed in a complicated fashion through multiple mergers and accretion along filaments. Still this very simple (and a very useful) prescription gives quite accurate predictions [15].

The following of this lecture is organized in the following way. Section 2 gives the equations which we solve to follow the evolution of initially small fluctuations. A brief discussion of different methods is given in section 3. Effects of the resolution and some other technical details are also discussed in Section 3. Identification of halos ("galaxies") is discussed in Section 4.

2. Equations of evolution of fluctuations in an expanding universe

Usually the problem of the formation and dynamics of cosmological objects is formulated as N-body problem: for N point-like objects with given initial positions and velocities find their positions and velocities at any consequent moment. It should be remembered that this just a short-cut in our formulation – just to make things simple and avoid any discussion. While it still mathematically correct in many cases, it does not explain what we are doing. If we are literally to take this approach, we should follow the motion of zillions of axions, baryons, neutrinos, and whatever else our Universe is made of. So, what it has to do with the motion of those few millions of particles in our simulations? The correct approach is to start with the Vlasov equation coupled with the Poisson equation (and proper initial and boundary conditions). If neglect the baryonic component, which of course is very interesting, but would complicate our situation too much, the system is described by distribution functions $f_i(\mathbf{x}, \dot{\mathbf{x}}, t)$ which should include all different clustered components i. For a simple CDM model we have only one component (axions or whatever it is). For more complicated Cold plus Hot Dark

Matter (CHDM) with few different types of neutrinos the system includes one DF for the cold component and one DF for each type of neutrino ([16]). The equations for the evolution of f_i are:

$$
\begin{aligned}
&\frac{\partial f_i}{\partial t} + \mathbf{x}\frac{\partial f_i}{\partial \mathbf{x}} - \nabla\phi\frac{\partial f_i}{\partial \dot{\mathbf{x}}} = 0, \quad \nabla^2\phi = 4\pi G a^2(\rho(\mathbf{x},t) - \rho_b(t)) = 4\pi G a^2 \Omega_{\rm dm}\delta_{\rm dm}, \\
&\delta_{\rm dm}(\mathbf{x},t) = (\rho_{\rm dm} - \langle\rho_{\rm dm}\rangle)/\langle\rho_{\rm dm}\rangle), \quad \rho_{\rm dm}(\mathbf{x},t) = a^{-3}\sum_i m_i \int d^3\dot{x} f_i(\mathbf{x},\dot{\mathbf{x}},t).
\end{aligned}
\tag{1}
$$

Here a is the expansion parameter, $\Omega_{\rm dm}$ is the contribution of the clustered dark matter to the mean density of the Universe, m_i is the mass of a particle of ith component of the dark matter. The solution of the Vlasov equation can be given in terms of characteristic equations, which *look* like equations of particle motion. The distribution function is constant along each characteristic. Complete set of characteristics is equivalent to the Vlasov equation. Of course, we can not have the complete set, but we can follow the evolution of the system (with some accuracy) if we select a representative sample of characteristics. One way of doing this would be to split initial phase space into small domains, take only one characteristic as representative for this volume element, and follow the evolution of the system of the "particles" in self-consistent way. In models with one "cold" component of clustering dark matter (like CDM or ΛCDM) the initial velocity is a unique function of coordinates (only "Zeldovich" part is present, no thermal velocities). This means that we need to split only coordinate space, not velocity space. For complicated models with significant thermal component (CHDM), the distribution in full phase space should be taken into account. Usually, this is simulated by placing 1-10 particles in each coordinate with velocities, which mimic the distribution of real "thermal" velocities. Depending on what we are interested in, we might split initial space into equal-size boxes (typical setup for PM or $\rm P^3M$ simulations) or we could divide some area of interest (say, where a cluster will form) into smaller boxes, and use much bigger boxes outside the area (to mimic gravitational forces of the outside material). In any case, the mass assigned to a "particle" is equal to the mass of the domain it represents. Now we can think of the "particle" either as a small box, which moves with the flow, but does not change its

original shape, or as a point-like particle. Both presentations are used in simulations. None is superior to another.

There are different forms of final equations. If we choose "momentum" $p = a^2\dot{x}$ as effective velocity ($v_{\rm pec} = p/a$,) and change the independent variable from time t to expansion parameter a, then the equations are

$$\begin{aligned} &\frac{d\mathbf{p}}{da} = -\frac{\nabla\phi}{\dot{a}}, \quad \frac{d\mathbf{x}}{da} = \frac{\mathbf{p}}{\dot{a}a^2}, \quad \nabla^2\phi = 4\pi G\Omega_0 a^2\delta_{\rm dm}, \\ &\dot{a} = \frac{H_0}{a^{1/2}}\sqrt{\Omega_0 + \Omega_{\rm curv,0}a + \Omega_{\Lambda,0}a^3}, \quad \Omega_0 + \Omega_{\rm curv,0} + \Omega_{\Lambda,0} = 1 \end{aligned} \tag{2}$$

where Ω_0, $\Omega_{\rm curv,0}$, and $\Omega_{\Lambda,0}$ are the density of the matter, effective densities of the curvature and cosmological constant in units of the critical density at $z = 0$. The curvature contribution is positive for negative curvature.

3. Methods

There are many different numerical techniques to follow the evolution of a system of many particles. For earlier reviews see Hockney & Eastwood [17] and Sellwood [18]. The most frequently used methods for cosmological applications fall in three classes: Particle Mesh (PM) codes, Particle-Particle/Particle-Mesh (P^3M) codes, and TREE codes. All methods have their advantages and disadvantages.

PM code. It uses a mesh to produce density and potential. As the result, its resolution is limited by the size of the mesh. Largest simulations were done by the author: 800^3 mesh with $3 \times 256^3 = 1.5 \times 10^8$ particles. New parallel supercomputer SP2 at Cornell will be used to run simulations with $1600^3 = 4.096 \times 10^9$ mesh. There are two advantages of the method: i) it is fast (the smallest number of operations per particle per time step of all the other methods), ii) it typically uses very large number of particles. The later can be crucial for some applications. There are few modifications of the code. There are few variants of PM code. "plain-vanilla" PM was described by Hockney & Eastwood [17]. It includes Cloud-In-Cell density assignment and 7-point discrete analog of the laplacian operator. Higher order

approximations improve the accuracy on large distances, but degrade the resolution (e.g. [19]). In an effort to reduce the order of approximation and to increase the resolution, Melott [20] introduced staggered mesh. It gives a better resolution on cell-size distances, but particles get self-forces (an isolated particle experiences a force from itself), which might be not a welcome feature. No serious testing of the method was done.

P^3M code is described in detail in [17] and [21]. It has two parts: PM part, which takes care of large-scale forces, and PP part, which adds small-scale particle-particle contribution. The simulations usually have 64^3–100^3 particles. The best known to me simulations were done by Ma & Bertschinger [22] for CHDM model: 128^3 of cold particles and ten times more hot particles. Because of strong clustering at late stages of evolution, PP part becomes prohibitively expensive once large objects start to form in large numbers. Significant speed is achieved in modified version of the code, which introduces subgrids (next levels of PM) in areas with high density [23]. With modification the code runs as fas as TREE code even for heavily clustered configurations [23].

TREE code is the most flexible code in the sense of the choice of boundary conditions [24, 25, 26. It is also more expensive than PM: it takes 10-50 times more operations. Bouchet & Hernquist [27] and Hernquist, Suto & Bouchet [28] extended the code for the periodical boundary conditions, which is important for simulating large-scale fluctuations. An interesting blend of PM and TREE code as done by Xu [29]. The code might be easier than usual to implement on parallel supercomputers.

Multigrid methods were introduced long ago, but only recently they started to show a potential to produce real results [30, 31, 32]. It worth of paying attention if a "multigrid" code is really a fully adaptive multigrid code. Some of the them are actually two-level mesh codes. Those codes provide increased resolution (by factor 4–8) inside one predefined region.

What code is the best? Which one to choose? Should we through away results of low resolution codes? (Tempting, but I would not do that). There is no unique answer –

everything depends on the problem, which we are addressing. For example, if we are interested in explanation of the large-scale structure (filaments, voids, Zeldovich approximation, and so on), PM code with 256^3 mesh is sufficient. It takes only one night to make a simulation on a (good) workstation. There is a very long list of problems like that. But if you intent to look for galaxies in the large-scale environment, much better resolution is needed. Sometimes a minimum dynamical range of 10^4 is quoted: we need to go to 10 kpc to see a galaxy, and we need to go up to at least 100 Mpc in order to have large voids and superclusters. What we should also include in this "wish-list" is a required mass range. If we would like to have something, which looks like a $10^{11}\ M_\odot$ galaxy, it should consist of many particles. Say, 30-100 particles. This gives $10^9\ M_\odot$ per particle and 410^3 particles for 100 Mpc box. All existing methods fall short of these requirements: PM is lacking resolution (a factor of ten), others do not have enough mass resolution (a factor of ten).

4. Distribution of the matter and effects of resolution

Figure 1 shows an example of structures observed in cosmological models. The following model with cosmological constant was chosen: $\Omega_0 = 0.30$, $h = 0.70$, $\sigma_8 = 1.1$, $Q_{\rm rms-PS} = 21.8\mu K$. The simulation was done with the PM code with 256^3 particles, 800^3 mesh, $\Delta a = 0.003$, box size $50h^{-1}$Mpc. Dark matter particles in a $1h^{-1}$Mpc slice are shown. Different structures can be identified on the plot. Large empty (of galaxies) voids, long filaments, few groups of galaxies, and many isolated galaxies. While it is not clear from this plot whether we actually see filaments or sheets, Kofman *et al.* [33] recently argued that they actually are filaments. It was a long-standing problem: the Zeldovich approximation was predicting "pancakes" and what was "numerically observed" looked more like filaments. The argument from the Zeldovich approximation is still valid. If we smooth initial density field and then randomly pick up a point at very early stage and trace its trajectory, then the particle will first hit a sheet, not a filament, because its middle eigenvalue of the deformation tensor is zero (on average) and the smallest one is negative. (Positive sign of an eigenvalue implies

collapse along corresponding direction. Only one positive eigenvalue means collapse along only one directions – a sheet forms). But Kofman *et al.* argue that this is not we are doing. We observe particle at a given moment. In this case we are asking what is the shape of an isodensity surface with density of few mean densities. This condition results in positive middle eigenvalue while keeping the expansion along the third axis – this is a filament.

In order to discuss objects on smaller scales we need to understand what happens with an isolated object, which size is close to the limit of resolution. In all codes the resolution (either a cell size or a softening parameter) is constant in *comoving* not proper coordinates. This means that the resolution actually decreases with time. (Look at the situation from a positive angle - the resolution gets much better if we go to high redshifts). For example, our resolution is 50 kpc and we have a galaxy with radius 50 kpc collapsing at $z = 4, a = 0.2$. The resolution at that redshift was 10 kpc – enough to reasonably resolve the object. Now, our resolution drops very severely. If this happened quickly (on dynamical scale), the galaxy would explode: its total energy would be positive. Fortunately, this does not happen because the time-scale of growth of the softening parameter is equal to the expansion rate of the Universe, and the later is always longer than the dynamical time. Thus the galaxy will adiabatically expand once the softening length gets close to its radius. Figure 2 illustrates this behavior. A system of 300 particles with initial velocities slightly below virial velocities was simulated using a PP code. Initially the particles were randomly distributed in a sphere with 100 kpc radius. Parameters were scaled in such a way that the total mass of the system is 10^{12} $M_\odot$, initial moment corresponds to $a = 0.2, z = 4$. Collapse happens at $a = 0.25, z = 3$. The radius of half mass is 50 kpc after collapse. The system was run once with softening parameter $\epsilon = 10$ kpc in *proper* coordinates, and another time with $\epsilon = 10$ kpc in *comoving* coordinates. Figure 2 shows snapshots of the simulations. Comoving coordinates are chosen to display the particles. Coordinate scale is in units of 100 kpc. Bottom row shows the evolution of the system simulated with constant proper resolution, while the top row is for constant comoving resolution. The circles show the size of resolution. Radius of each circle is 1.5ϵ. At this radius

the error in force at 1 cell radius in PM code is equal to the error in PP code. As time goes on the system shrinks in comoving coordinates. But once its size is close to the resolution, the shrinking stops.

This just confirms what is found in numerical simulations: isolated unresolved "galaxies" are small, almost spherical balls with 2 grid cells across their diameters.

In order get a better insight on the effects of the resolution and to understand what should be expected if we significantly increase the resolution, two simulations were run with exactly the same initial conditions, but with twice different resolution. The same cosmological model as before was chosen (ΛCDM with $\Omega_0 = 0.3, h = 0.70$). PM code with 128^3 particles, mesh 450^3 and 216^3, box size $20h^{-1}$Mpc was used. This gives the resolution of $44h^{-1}$kpc and $92h^{-1}$kpc for the two runs and mass for a particle $m_1 = 3.1 \times 10^8 h^{-1}\ M_\odot$. Figure 3 shows particles in a small 3x3 Mpc window in the simulations at $z = 3$. Only particles with estimated density indicated in the figure are shown and only 1/4 of all particles is displayed. Because the same isolated clump would have smaller radius and, thus, higher density with twice better resolution, density limits were slightly adjusted to take into account the difference in the resolution. Figure 4 shows the same window at $z = 0$. While plots for higher resolution show more small clumps, all large clumps are found in both plots. The differences are as expected: higher resolution results in more compact and dense objects. The most significant result lies in what was feared, but not found. The largest object in $z = 0$ plot has radius of about 300 kpc. The usual question was: having a low-resolution run with 100 kpc resolution, how do you know that with better resolution it will not split into, say into two objects? Our results indicate that it does not happen: with more then twice better resolution the object got a bit smaller, a bit rounder, got few tiny satellites, which were barely seeing with the low resolution, but it does not show any tendency to break int large peaces. Figures 5 and 6 show another examples of comparison of the two simulations. In this case we deal with a group of galaxies. The result is just the same: more small objects, the same large objects, no tendency for splitting of large halos into smaller ones.

While visual analysis indicates that we might be quite ok with relatively low resolution if only large halos (above 10^{10} $M_\odot$) are considered, it is not that easy to make quantitative analysis. Comparison of coordinates, velocities, and densities of individual particles in both simulations would be a natural test, but it just fails. The reason is the divergence of trajectories in dynamical systems: small differences in initial coordinates result in large differences after few dynamical times. The winding problem of the spiral pattern is just one of the manifestation of this divergence. As the result, deviations of coordinates $x_{44} - x_{92}$ and velocities $V_{x,44} - V_{x,92}$ as functions of density (Figures 7 and 8) indicate mainly sizes of virialized object, which exist at different redshift. Even in spite of the divergence of the trajectories, extremely large fraction of particles – 99% – indicate the difference in coordinates less than 2 cell sizes and difference in velocities less than 100 km/s.

5. Halo identification and overmerging problem

There are different methods of identifying collapsed objects (halos) in numerical simulations.

Friends-Of-Friends (FOF) algorithm was used a lot and still has its adepts. If we imagine that each particle is surrounded by a sphere of radius $bd/2$, then every connected group of particles is identified as a halo. Here d is the mean distance between particles, and b is called *linking parameter*, which typically is 0.2. Dependance of groups on b is extremely strong. The method stems from an old idea to use percolation theory to discriminate between cosmological models. Because of that, FOF is also called percolation method, which is wrong because the percolation is about groups spanning the whole box, not collapsed and compact objects. FOF was criticized for failing to find separate groups in cases when those groups were obviously present [19]. The problem originates from the tendency of FOF to "percolate" through bridges connecting interacting galaxies or galaxies in high density backgrounds.

DENMAX tried to overcome the problems of FOF by dealing with density maxima [19,34]. It finds maxima of density and then tries to identify particles, which belong to each

maximum (halo). The procedure is quite complicated. First, density field is constructed. Second, the density (with negative sign) is treated as potential in which particles start to move as in a viscous fluid. Eventially, particles sink at bottoms of the potential (which are also maxima density). Third, only particles with negative energy (relative to their group) are retained. Just as in the case of FOF, we can easily imagine situations when (this time) DENMAX should fail. For example, two colliding galaxies in a cluster of galaxies. Because of large relative velocity they should just pass each other. In the moment of collision DENMAX ceases to "see" both galaxies because all particle have positive energies. That is probably quite unlikely situation. The method is definitely one of the best at present. The only problem is that it seems to be too complicated for present state of simulations.

"Overdensity 200". There is no name for the method, but it is often used. Find density maximum, place a sphere and find radius, within which the sphere has the mean overdensity 200 (or 177 if you really want to follow the top-hat model of nonlinear collapse). In Figure 5 all particles in the central $\sim$1 Mpc area will be just one halo. The mass function of dark halos, constructed in this way, has a very long tail extending into masses, which are far too large for individual galaxies. The existence of this tail is often called "overmerging problem". Figure 9 gives comparison of mass functions of dark halos $n(> M)$ in previous two simulations with different resolutions. The full curve shows the mass function for high resolution run. Note that high mass tail of the mass functions does not depend on resolution, which indicates convergence of the results. The mass function flattens in both simulations at different level (more small-mass objects with high resolution), but at the same mass, which corresponds to 20-30 particles per object.

Actually, there are two "overmerging" problems. One arises because of naive application of the top-hat model. Consider our Galaxy and Andromeda Nebula as a testbed. If we assume that both galaxies have flat rotation curves at least up to the radius of 100 kpc (which might be true), then overdensity at 100 kpc is about 4000 for our Galaxy and about twice that for Andromeda Nebula. The overdensity at half way between galaxies is about 200–300. Thus,

if we had fantastic simulation with our Galaxy and M31 as they are in the Universe, our "overdensity 200" method would lump them together. There are different ways of fixing the problem. DENMAX probably would find both galaxies.

Unfortunately, there is real "overmerger" problem, which cannot be avoided that easily. For a long time it was assumed that appearance of extremely large halos ("overmergers") is due to insufficient resolution. Numerical simulations indicated that with smaller and smaller mass of each particle, and with better and better resolution we see more and more substructure in overmergers. But it seems that the tendency have stopped. Now extra resolution does not split large overmergers. The problem is real and it is not a numerical artifact, which can be cured by much better resolution. Figures 3-6 illustrate this result. Recently, Moore *et al.* [35] and van Kampen [36] gave a very plausible explanation: tidal forces. If a small halo ("galaxy") falls into a larger one ("group"), it will be tidally disrupted at a distance where its central density is equal to the mean density of the large halo at that distance (for "group" with $\rho \propto r^{-2}$). Because the core of the large halo likely was formed earlier than the small halo, it has higher density. The only chance for the small halo to survive is to stay in peripheral parts of the group. Numerical simulations show that this is what happens.

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Figure Captions

Figure 1 An example of structures observed in cosmological models with cosmological constant: $\Omega_0 = 0.30$, $h = 0.70$, $\sigma_8 = 1.1$, $Q_{\rm rms-PS} = 21.8\mu K$. The simulation was done with the PM code with 256^3 particles, 800^3 mesh. Dark matter particles in a $1h^{-1}$Mpc slice are shown.

Figure 2 Evolution of a system of 300 particles in comoving coordinates. Initially the particles were randomly distributed in a sphere with 100 kpc radius. Collapse happens at $a = 0.25, z = 3$. The system was run once with softening parameter $\epsilon = 10$ kpc in *proper* coordinates (bottom row), and another time with $\epsilon = 10$ kpc in *comoving* coordinates (top row). Coordinate scale is in units of 100 kpc. The circles show the size of resolution. Radius of each circle is 1.5ϵ.

Figure 3 Particles in a small 3x3 Mpc window at $z = 3$ in ΛCDM model with $\Omega_0 = 0.3, h = 0.70$. Only particles with estimated density indicated in the figure are shown and only 1/4 of all particles is displayed.

Figure 4 The same window at $z = 0$.

Figure 5 An example of a group of galaxies simulated with with different resolutions.

Figure 6 The sane as in Figure 5, but with different density threshold.

Figure 7 Deviations of coordinates and velocities in high and low resolution simulations at $z = 3$.

Figure 8 The same as in Figure 7, but at $z = 0$.

Figure 9 Comparison of mass functions of dark halos $n(> M)$ in two simulations with different reolutions ($44h^{-1}$ kpc and $92h^{-1}$ kpc). The full curve shows the mass function for high resolution run. Note that high mass tail of the mass functions does not depend on resolution, which indicates convergence of the results.